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A method is presented for solving systems of generalized Abel integral equations by reduction to equivalent coupled Riemann-Hilbert boundary value problems. It is demonstrated how the Riemann system may be uncoupled and closed form solutions computed. An application is then given to solving certain simultaneous dual relations which arise in bimedia fracture problems in elasticity.

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Systems of Generalized Abel Integral Equations

by

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1. Introduction

As demonstrated in Lowengrub [3], certain mixed boundary value problems arising in the classical theory of elasticity reduce to the problem of determining functions φ_1 and φ_2 satisfying Abel type integral equations of the type,

$$\alpha(x)\int_0^x \frac{\varphi_1(t)\,dt}{(x^\rho - t^\rho)^\mu} + \beta(x)\int_x^1 \frac{\varphi_2(t)\,dt}{(t^\rho - x^\rho)^\mu} = h_1(x), \qquad a<x<b \tag{1.1}$$

$$\gamma(x)\int_x^1 \frac{\varphi_1(t)\,dt}{(t^\rho - x^\rho)^\mu} + \delta(x)\int_0^x \frac{\varphi_2(t)\,dt}{(x^\rho - t^\rho)^\mu} = h_2(x), \qquad a<x<b \tag{1.2}$$

where $0<\mu<1$, $\rho \ge 1$ and the functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$ and $\delta(x)$ have derivatives satisfying Hölder conditions on (a,b). It is also assumed that $h_i(x)$, $i=1,2$, is Holder continuous on the interval.

In this paper we show that systems of the type (1.1) and (1.2) can be reduced to the determination of a matrix function $\Phi(z) = (\Phi_1(z), \Phi_2(z))$ analytic in the plane cut along (a,b), satisfying certain growth conditions at ∞ and along the cut, (a,b),

$$A(x)\Phi^+(x) = -e^{\mu\pi i}\overline{A(x)}\Phi^-(x) + F(x) \tag{1.3}$$

where $A(x)$ is a coefficient matrix with elements linear

combinations of α,β,γ and δ. The system (1.3) is a coupled system. We effect a linear uncoupling of this system by introducing certain similarity transformations. The matrices associated with these transformations are explicitly computed and hence exact solutions are derived. In physical applications, such as the determination of the stress field in an inhomogeneous body containing flaws, the relevant physical quantities are expressed in terms of the matrix function $\Phi(z)$. One need not actually solve for φ_1 and φ_2 in (1.1) and (1.2). The functions $\Phi_1(z)$ and $\Phi_2(z)$ are defined (say in the case $\rho = 1$) by

$$\Phi_1(z) = \frac{1}{R(z)} \int_a^b \frac{\varphi_i(t)dt}{(t-z)^{\mu}}$$

where $R(z) = [(z-a)(b-z)]^{\frac{1-\mu}{2}}$. These functions must be defined on appropriate branches. Analogous representations are introduced for $\rho \geq 1$.

Section 2 of the paper thoroughly analyzes the case $\rho = 1$ while in section 3 we choose $\rho = 2$. These are the two cases of physical interest. In section 4 we consider some explicit examples: (i) $\alpha(x) = \alpha$, $\beta(x) = \beta$, $\gamma(x) = \gamma$ and $\delta(x) = \delta$ with $\mu = \frac{1}{2}$, $\rho = 1$ and α,β,γ and δ constant; (ii) $\alpha(x) = \beta$, $\beta(x) = \frac{\alpha}{x}$, $\gamma(x) = \alpha$ and $\delta(x) = -\frac{\beta}{x}$ with $\mu = \frac{1}{2}$, $\rho = 2$, and α,

β constant. The final section demonstrates how general dual relations (given in terms of the Erdelyi-Sneddon operators of fractional integration - see Erdelyi-Sneddon [1]) may be reduced to systems of the type (1.1) and (1.2).

2. The first generalized Abel system.

In this section we consider the generalized Abel system of equations,

$$\alpha(x)\int_a^x \frac{\varphi_1(t)\,dt}{(x-t)^{\mu}} + \beta(x)\int_x^b \frac{\varphi_2(t)\,dt}{(t-x)^{\mu}} = f_1(x),\ x \in (a,b) \tag{2.1}$$

$$\gamma(x)\int_x^b \frac{\varphi_1(t)\,dt}{(t-x)^{\mu}} + \delta(x)\int_a^x \frac{\varphi_2(t)\,dt}{(x-t)^{\mu}} = f_2(x),\ x \in (a,b) \tag{2.2}$$

where α, β, γ, and δ satisfy conditions to be specified later. However, we do assume that f_1 and f_2 are Hölder continuous on (a,b).

As in Sakalyuk [6], we define the sectionally analytic functions

$$\Phi_i(z) = \frac{1}{R(z)}\int_a^b \frac{\varphi_i(t)}{(t-z)^{\mu}}\,dt, \qquad i = 1,2. \tag{2.3}$$

where $R(z) = [(z-a)(b-z)]^{\frac{1-\mu}{2}}$ and the function is defined by some branch. If $\varphi_i^*(t)$ satisfies $\varphi_i(t) = \varphi_i(t)[(t-a)(b-t)]^{\mu+\epsilon-1}$ where $\epsilon > 0$ and $\varphi_i^*(t)$ is Hölder on $[a,b]$, then $\Phi_i(z)$ is analytic in the plane cut along $[a,b]$. Moreover, the boundary limits $\Phi_i^{\pm}(x)$, where

$$\Phi_i^{+}(x) = \lim_{\substack{z \to x \\ \mathrm{Im}(z) > 0}} \Phi_i(z) \qquad a < x < b$$

and

$$\Phi_i^{-}(x) = \lim_{\substack{z \to x \\ \mathrm{Im}(z) < 0}} \Phi_i(z), \qquad a < x < b$$

are continuous. In addition,

$$\Phi_i(z) = O(|z-a|^{\frac{\mu-1}{2}}) , \qquad z \to a$$

$$\Phi_i(z) = O(|z-b|^{\frac{\mu-1}{2}}) , \qquad z \to b \tag{2.4}$$

$$\Phi_i(z) = O(|z|^{-1}) , \qquad z \to \infty .$$

A simple calculation verifies that

$$\Phi_i^+(x) = \frac{1}{R(x)}\left[e^{\mu\pi i}\int_\alpha^x \frac{\varphi_i(t)dt}{(x-t)^\mu} + \int_x^b \frac{\varphi_i(t)dt}{(t-x)^\mu}\right] \tag{2.5}$$

$x \in (a,b)$

and

$$\Phi_i^-(x) = -\frac{1}{R(x)}\left[\int_a^x \frac{\varphi_i(t)dt}{(x-t)^\mu} + e^{\mu\pi i}\int_x^b \frac{\varphi_i(t)dt}{(t-x)^\mu}\right] . \tag{2.6}$$

It immediately follows that

$$\int_a^x \frac{\varphi_i(t)dt}{(x-t)^\mu} = \left[\frac{e^{\mu\pi i}\Phi_i^+(x) + \Phi_i^-(x)}{e^{2\mu\pi i} - 1}\right]R(x), \tag{2.7}$$

$x \in (a,b)$

and

$$\int_x^b \frac{\varphi_i(t)dt}{(t-x)^\mu} = -\left[\frac{\Phi_i^+(x) + e^{\mu\pi i}\Phi_i^-(x)}{e^{2\mu\pi i} - 1}\right]R(x) , \tag{2.8}$$

$i = 1,2$.

Substitution of (2.7) and (2.8) into (2.1) and (2.2) yields

the boundary condition on $a < x < b$,

$$\alpha(x)e^{\mu\pi i}\Phi_1^+(x) - \beta(x)\Phi_2^-(x) + \alpha(x)\Phi_1^-(x) - \beta(x)e^{\mu\pi i}\Phi_2^-(x) = F_1(x) \quad (2.9)$$

$$-\gamma(x)\Phi_1^+(x)+\delta(x)e^{\mu\pi i}\Phi_2^+(x)-\gamma(x)e^{\mu\pi i}\Phi_1^-(x)+\delta(x)\Phi_2^-(x) = F_2(x) \quad (2.10)$$

where

$$F_i(x) = f_i(x)(e^{2\mu\pi i} - 1)/R(x) .$$

This substitution then reduces our problem to determining two sectionally analytic functions $\Phi_1(z)$, $\Phi_2(z)$ satisfying the growth conditions (2.4); that is, solving a coupled Riemann-Hilbert boundary value problem. Once we have determined $\Phi_j(z)$, $(i = 1,2)$, the functions φ_1 and φ_2 are obtained by solving the Abel integral equations (2.7) and (2.8).

For convenience we introduce the following matrix notation; set,

$$\Phi(z) = (\Phi_1(z), \Phi_2(z))^T \quad \text{and} \quad F(x) = (F_1(x), F_2(x))^T$$

so that the boundary conditions (2.9) and (2,10) may be written in the form

$$A(x)\Phi^+(x) = - e^{\mu\pi i}\overline{A(x)}\Phi^-(x) + F(x) , \qquad a < x < b \qquad (2.11)$$

where $A(x) = (a_{ij}(x))$, $i,j = 1,2$ with

$$a_{11}(x) = \alpha(x)e^{\mu\pi i}, \quad a_{12}(x) = -\beta(x), \quad a_{21}(x) = -\gamma(x) \text{ and} \qquad (2.11a)$$

$$a_{22}(x) = \delta(x)e^{\mu\pi i}.$$

If we require the condition

$$\det A(x) = \alpha(x)\delta(x)e^{2\mu\pi i} - \gamma(x)\beta(x) \neq 0, \quad a < x < b \tag{2.12}$$

then the matrix $A(x)$ is invertible and (2.11) is equivalent to the boundary condition

$$\Phi^+(x) = -e^{\mu\pi i}G(x)\Phi^-(x) + g(x), \qquad a < x < b \tag{2.13}$$

where

$$G(x) = A^{-1}(x)\overline{A(x)} \quad \text{and} \quad g(x) = A^{-1}(x)F(x). \tag{2.14}$$

It is necessary for us to determine conditions whereby the coupled Riemann-Hilbert problem can be uncoupled. We shall effect a linear uncoupling of the system (2.9) and (2.10) by finding a non-singular matrix $P(z)$, analytic in the complex plane (except for perhaps a finite number of poles) with a pole at infinity and such that for $a < x < b$

$$P(x)G(x)P^{-1}(x) = D(x) = (d_{ij}(x))$$

with $d_{12}(x) = d_{21}(x) = 0$.

Let $\Sigma(z) = (\Sigma_1(z), \Sigma_2(z))^T$ be defined as follows:

$$\Sigma(z) = P(z)\Phi(z). \tag{2.15}$$

Note that $\Sigma(z)$ is analytic in the plane (except for perhaps a finite number of poles) cut along $a < x < b$. In addition, $\Sigma(z)$ satisfies appropriate growth conditions at infinity.

Substitution of (2.14) into the boundary condition (2.13) yields the uncoupled set of conditions,

$$\Sigma_i^+(x) = -e^{\mu\pi i} d_{ii}(x) \Sigma_i^-(x) + k_i(x), \quad a < x < b \tag{2.16}$$

$i = 1,2$ and $k(x) = P(x)g(x)$. Thus, this procedure reduces our problem to the determination of two sectionally analytic functions $\Sigma_1(z)$ and $\Sigma_2(z)$ satisfying appropriate growth conditions at infinity and the boundary conditions (2.15). The solution to these Riemann Hilbert problems is well-known (once the index is determined) (see for example, Muskhelishvili [5]).

The main problem is to explicitly determine the matrix P. We first compute $G(x)$ from (2.14) .

$$G(x) = [\det A(x)]^{-1} T(x)$$

where $T(x) = [t_{ij}(x)]$ with

$$t_{11} = t_{22} = \alpha(x)\delta(x) - \beta(x)\gamma(x), \quad t_{12} = -2i\beta(x)\delta(x)\sin\mu\pi$$

and $t_{21} = -2i\alpha(x)\gamma(x)\sin\mu\pi$.

A non-singular matrix, $P(x)$, for which $P(x)T(x)P^{-1}(x)$ is diagonal exists if and only if $T(x)$ has two linearly independent eigenvectors. In this case, the matrix $P(x)T(x)P^{-1}(x)$ has the eigenvalues of $T(x)$ as its diagonal elements and $P(x)$ has as its rows two independent eigenvectors of $T(x)$. If $t_{12}(x)t_{21}(x) \neq 0$ then the eigenvalues of $T(x)$ are $t_{11}(x) \pm \sqrt{t_{12}(x)t_{21}(x)}$ so that for the matrix $P(x)$ we may take

$$P(x) = c(x)\begin{bmatrix} t_{12}(x) & -\sqrt{t_{12}(x)t_{21}(x)} \\ t_{12}(x) & \sqrt{t_{12}(x)t_{21}(x)} \end{bmatrix} \tag{2.17}$$

where $c(x)$ is any <u>scalar</u> function.

If $t_{12}(x)t_{21}(x) = 0$, then $T(x)$ has two independent eigenvectors if and only if both $t_{12}(x)$ and $t_{21}(x)$ vanish. Since $T(x)$ is non-singular this can occur if and only if either $\delta(x) = \alpha(x) = 0$ and $\gamma(x)\beta(x) \neq 0$ or $\gamma(x) = \beta(x) = 0$ and $\delta(x)\alpha(x) \neq 0$. In this case, $T(x)$ is just a scalar multiple of the identity. For most applications, $t_{12}(x)t_{21}(x) \neq 0$ except perhaps at the endpoints $x = a$ and $x = b$. Such exceptional

cases are easily handled. In order to simplify further work, we assume that $t_{12}(x)t_{21}(x) \neq 0$.

Let $P(x)$ be defined by (2.17) so that

$$P(x)T(x)P^{-1}(x) = \begin{pmatrix} t_{11}(x) + \sqrt{t_{12}(x)t_{21}(x)} & 0 \\ 0 & t_{11}(x) - \sqrt{t_{12}(x)t_{21}(x)} \end{pmatrix}$$

which provides the desired uncoupling of the pair of coupled Riemann-Hilbert boundary value problems. The scalar $c(x)$ is chosen so that $P(x)$ can be extended to a matrix $P(z)$ meromorphic in the plane with a pole at infinity. This enables us to explicitly determine Σ_1 and Σ_2 and hence Φ_1 and Φ_2. Inversion of (2.7), (2.8) yields our original unknown functions φ_1 and φ_2. An example of this analysis appears in section 4.

3. <u>The second generalized Abel system</u>.

We next consider the system

$$\alpha(x)\int_0^x \frac{\varphi_1(t)dt}{(x^2-t^2)^\mu} + \beta(x)\int_x^1 \frac{\varphi_2(t)dt}{(t^2-x^2)^\mu} = f_1(x) , \qquad 0<x<1 \quad (3.1)$$

$$\gamma(x)\int_x^1 \frac{\varphi_1(t)dt}{(t^2-x^2)^\mu} + \delta(x)\int_0^x \frac{\varphi_2(t)dt}{(x^2-t^2)^\mu} = f_2(x) , \qquad 0<x<1. \quad (3.2)$$

The interval (0,1) is chosen rather than (a,b) in order to simplify the analysis. It is trivial to extend to (a,b). The case $\mu = 1/2$ in (3.1) and (3.2) has been considered by Lowengrub [3].

Analogous to the method used in section 2, we introduce the sectionally analytic functions $\Phi_1(z)$ and $\Phi_2(z)$ defined by,

$$\Phi_i(z) = (z^2 - 1)^{\mu-\frac{1}{2}} \int_0^1 \frac{\varphi_i(t)dt}{(z^2-t^2)^{\mu}} , \quad i = 1,2 \tag{3.3}$$

If $\varphi_i(t)$ satisfies $\varphi_i(t) = \varphi_i^*(t) t^{\mu} [t(1-t)]^{\mu+\epsilon-1}$ where $\epsilon = 0$ and $\varphi_i^*(t)$ is Holder continuous on [0,1], then $\Phi_i(z)$ $(i = 1,2)$ is analytic in the plane cut along [-1,1] and satisfies the following conditions,

$$\Phi_i(z) = O(|z-1|^{\mu-\frac{1}{2}}) \quad \text{as} \quad z \to 1$$

$$\Phi_i(z) = O(|z+1|^{\mu-\frac{1}{2}}) \quad \text{as} \quad z \to -1 \tag{3.4}$$

and

$$\Phi_i(z) = O(|z|^{-1}) \quad \text{as} \quad z \to \infty .$$

Moreover, the limiting values $\Phi_i^{\pm}(x)$ are continuous functions for $|x| < 1$ except perhaps for $x = 0$.

For each of the functions $(z-1)^{\mu-\frac{1}{2}}$, $(z+1)^{\mu-\frac{1}{2}}$, $(z-t)^{-\mu}$ and $(z+t)^{-\mu}$ we take as the branch cut that line lying along the positive x-axis and restrict their arguments to lie between 0 and 2π. The following limits, for $0 < x < 1$, are easily computed:

$$\Phi_i^+(x) = -i(1-x^2)^{\mu-\frac{1}{2}}\left\{\int_x^1 \frac{\varphi_i(t)dt}{(t^2-x^2)^\mu} + e^{\mu\pi i}\int_0^x \frac{\varphi_i(t)dt}{(x^2-t^2)^\mu}\right\} \tag{3.5}$$

$$\Phi_i^-(x) = i(1-x^2)^{\mu-\frac{1}{2}}\left\{\int_x^1 \frac{\varphi_i(t)dt}{(t^2-x^2)^\mu} + e^{-\mu\pi i}\int_0^x \frac{\varphi_i(t)dt}{(x^2-t^2)^\mu}\right\}; \tag{3.6}$$

whereas for $-1 < x < 0$

$$\Phi_i^+(x) = -i(1-x^2)^{\mu-\frac{1}{2}}\left\{\int_{|x|}^1 \frac{\varphi_i(t)dt}{(t^2-x^2)^\mu} + e^{\mu\pi i}\int_0^{|x|} \frac{\varphi_i(t)dt}{(x^2-t^2)^\mu}\right\} \tag{3.7}$$

$$\Phi_i^-(x) = i(1-x^2)^{\mu-\frac{1}{2}}\left\{\int_{|x|}^1 \frac{\varphi_i(t)dt}{(t^2-x^2)^\mu} + e^{\mu\pi i}\int_0^{|x|} \frac{\varphi_i(t)dt}{(x^2-t^2)^\mu}\right\}. \tag{3.9}$$

It should be observed that for $-1 < x < 0$,

$$\Phi_i^\pm(x) = -\overline{\Phi_i^\pm(-x)}\ .$$

$$\Phi_i^+(x) + \Phi_i^-(x) = 2\sin\mu\pi(1-x^2)^{\mu-\frac{1}{2}}\int_0^x \frac{\varphi_i(t)dt}{(x^2-t^2)^\mu}\ , \tag{3.10}$$

and

$$e^{-\mu\pi i}\Phi_i^+(x) + e^{\mu\pi i}\Phi_i^-(x) = -2\sin\mu\pi(1-x^2)^{\mu-\frac{1}{2}}\int_x^1 \frac{\varphi_i(t)dt}{(t^2-x^2)^\mu} \tag{3.11}$$

for $0 < x < 1$.

Substitution of the above into the original set of integral equations (3.1) and 3.2) reduces the problem to the following:

determine two sectionally analytic functions $\Phi_1(z)$ and $\Phi_2(z)$ satisfying the conditions (3.4) and the boundary values,

$$[\alpha(x)\Phi_1^+(x)-\beta(x)e^{-\mu\pi i}\Phi_2^+(x)]+[\alpha(x)\Phi_1^- -\beta(x)e^{\mu\pi i}\Phi_2^-(x)]=F_1(x) \quad (3.12)$$

$$0<x<1$$

$$[-\gamma(x)e^{-\mu\pi i}\Phi_1^+(x)+\delta(x)\Phi_2^+(x)]+[\delta(x)\Phi_2^-(x)-\gamma(x)e^{\mu\pi i}\Phi_1^-(x)]=F_2(x) \quad (3.13)$$

$$0<x<1$$

where $\qquad F_i(x) = 2\sin\mu\pi(1-x^2)^{\mu-\frac{1}{2}}f_i(x)$.

In matrix notation the system (3.12)-(3.13) becomes

$$A(x)\Phi^+(x) = -B(x)\Phi^-(x)+F(x) \qquad 0<x<1 \qquad (3.14)$$

where $\Phi(z) = (\Phi_1(z),\Phi_2(z))^T$, $F(x) = (F_1(x),F_2(x))^T$,

$A(x) = (a_{ij}(x))$ with $a_{11}(x) = \alpha(x)$,

$a_{12}(x) = -\beta(x)e^{-\mu\pi i}$, $a_{21}(x) = -\gamma(x)e^{-\mu\pi i}$ and $a_{22}(x) = \delta(x)$,

and $\quad B(x) = \overline{A(x)}$. $\qquad$ (3.14a)

In order to determine $\Phi(z)$, boundary conditions for $\Phi^{\pm}(x)$ must be extended to all of $[-1,1]$. It is clear from (3.9) how this extension is to be performed.

In particular, we obtain the system

$$\hat{A}(x)\Phi^+(x) = -\hat{B}(x)\Phi(x)+\hat{F}(x) \qquad -1<x<1 \qquad (3.15)$$

where $$\hat{A}(x) = \begin{cases} A(x) & 0<x<1 \\ \overline{-A(-x)} & -1<x<0 , \end{cases}$$

$$\hat{B}(x) = \begin{cases} B(x) & 0<x<1 \\ \overline{-B(-x)} & -1<x<0 , \end{cases} \qquad (3.16)$$

and $$\hat{F}(x) = \begin{cases} F(x) & 0 < x < 1 \\ \overline{F(-x)} & -1 < x < 0 . \end{cases}$$

As in section 1, we assume that $\hat{A}(x)$ is invertible for $-1 < x < 1$. The expressions in (3.16) imply that it suffices to assume that $A(x)$ is invertible for $0 < x < 1$, or equivalently, that

$$\alpha(x)\delta(x)e^{2\mu\pi i} - \gamma(x)\beta(x) \neq 0 \qquad 0 < x < 1 .$$

This gives us the system,

$$\Phi^{+}(x) = -\hat{G}(x)\Phi^{-}(x) + \hat{g}(x) \qquad -1 < x < 1 \qquad (3.17)$$

where $\hat{G}(x) = \hat{A}^{-1}(x)\hat{B}(x) = \hat{A}^{-1}(x)\overline{\hat{A}(x)}$

and $\hat{g}(x) = \hat{A}^{-1}(x)\hat{F}(x)$.

Thus, we must now find a sectionally analytic matrix, $\Phi(z)$ satisfying appropriate conditions at 1 and -1, a growth condition $\Phi(z) = O(|z|^{-1})$ at infinity, and the boundary values (3.17). We seek a linear uncoupling of (3.17); that is, a non-singular matrix $\hat{P}(x)$ such that $\hat{P}(x)\hat{G}(x)\hat{P}^{-1}(x)$ is a diagonal matrix for $-1 < x < 1$.

We first compute $\hat{G}(x)$ on $0 < x < 1$ and obtain,

$$\hat{G}(x) = [\det(A(x))]^{-1}T(x)$$

where $T(x) = [t_{ij}(x)]$ with $t_{11}(x) = t_{22}(x) = \delta(x)\alpha(x) - \gamma(x)\beta(x)$, $t_{12}(x) = -2i\sin\mu\pi\delta(x)\beta(x)$ and $t_{21}(x) = -2i\sin\mu\pi\alpha(x)\gamma(x)$.

If we employ the arguments of section 1, we see that,

$$P(x) = c(x)\begin{pmatrix} t_{12}(x) & -\sqrt{t_{12}(x)t_{21}(x)} \\ t_{12}(x) & \sqrt{t_{12}(x)t_{21}(x)} \end{pmatrix} . \qquad (3.18)$$

$$= + 2i \sin \mu\pi \; c(x)\begin{pmatrix} -\delta(x)\beta(x) & -\sqrt{\alpha(x)\beta(x)\gamma(x)\delta(x)} \\ -\delta(x)\beta(x) & \sqrt{\alpha(x)\beta(x)\gamma(x)\delta(x)} \end{pmatrix}$$

(provided $t_{12}(x)t_{21}(x) \neq 0$)

produces the desired uncoupling of the boundary conditions (3.17) for $0 < x < 1$.

In particular,

$$P(x)T(x)P^{-1}(x) = \begin{pmatrix} t_{11}(x) + \sqrt{t_{12}(x)t_{21}(x)} & 0 \\ 0 & t_{11}(x) - \sqrt{t_{12}(x)t_{21}(x)} \end{pmatrix} .$$

on $0 < x < 1$.

Next we compute $\hat{G}(x)$ for $-1 < x < 0$. Since $\hat{G}(x) = \overline{\hat{G}(-x)}$ for $-1 < x < 0$, we find that

$$\hat{G}(x) = [\det(\overline{A(-x)})]^{-1}\overline{T(-x)} \qquad -1 < x < 0.$$

Moreover, since $\alpha(x), \beta(x), \gamma(x)$ and $\delta(x)$ are real, the rows of $P(x)$ are also independent eigenvectors of $\overline{T(x)}$ for $0 < x < 1$. Hence $P(-x)T(-x)P(-x)^{-1}$ is diagonal for $-1 < x < 0$, and if $\hat{P}(x)$ is defined by

$$\hat{P}(x) = \begin{cases} P(x) & 0 < x < 1 \\ P(-x) & -1 < x < 0 \end{cases} \quad \text{or} \quad \begin{cases} P(x) & 0 < x < 1 \\ \overline{P(-x)} & -1 < x < 0 \end{cases}$$

then $\hat{P}(x)\hat{G}(x)\hat{P}^{-1}(x)$ is diagonal for $-1 < x < 1$. If in addition

$\hat{P}(x)$ can be extended to a function $\hat{P}(z)$ meromorphic in $\not{C}$ with a pole at infinity, then $\Sigma(z) = \hat{P}(z)\,\Phi(z)$ defines a sectionally analytic function $\Sigma(z)$, satisfying properties analogous to 3.4 and the boundary condition

$$\Sigma^+(x) = -\hat{D}(x)\ \Sigma^-(x) + \hat{k}(x) \qquad -1 < x < 1 \tag{3.19}$$

where
$$\hat{D}(x) = \hat{P}(x)\hat{G}(x)\hat{P}^{-1}(x)$$
and $\hat{k}(x) = \hat{P}(x)\hat{g}(x)$. An example illustrating this analysis appears in the fourth section.

§4 Examples.

As a first example, we consider the set of equations (2.1), (2.2) with $\alpha(x) = \alpha$, $\beta(x) = \beta$, $\gamma(x) = \gamma$ and $\delta(x) = \delta$. Here, α, β, γ, and δ are all complex constants. Since this example appears in various applications, we display the appropriate matrices and write out the solution. We assume that $\alpha\beta\gamma\delta \neq 0$.

The coefficient matrices needed in (2.13) and (2.14) are given by:

$$A^{-1}(x) = \frac{1}{\alpha\delta e^{2\mu\pi i} - \gamma\beta} \begin{bmatrix} \delta e^{\mu\pi i} & \beta \\ \alpha & \alpha e^{\mu\pi i} \end{bmatrix}, \tag{4.1}$$

$$G(x) = \frac{1}{\alpha\delta e^{2\mu\pi i} - \gamma\beta}\begin{bmatrix} \alpha\delta - \beta\gamma & -\ 2i\beta\delta \sin\mu\pi \\ -2i\alpha\gamma \sin\mu\pi & \alpha\delta - \beta\gamma \end{bmatrix}, \tag{4.2}$$

$$P(x) = -2i\sin\mu\pi \begin{bmatrix} \delta\beta & \sqrt{\alpha\beta\gamma\delta} \\ \delta\beta & -\sqrt{\alpha\beta\gamma\delta} \end{bmatrix} \tag{4.3}$$

while the diagonal matrix, $D(x)$, used in (2.15) takes the form,

$$D(x) = \frac{1}{\alpha\delta e^{2\mu\pi i} - \gamma\beta}\begin{bmatrix} \alpha\delta - \beta\gamma + 2i\sin\mu\pi\sqrt{\alpha\beta\gamma\delta} & 0 \\ 0 & \alpha\delta - \beta\gamma - 2i\sin\mu\pi\sqrt{\alpha\beta\gamma\delta} \end{bmatrix}. \tag{4.4}$$

The problem is then reduced to the solution of the uncoupled Riemann Hilbert problem: determine $\Sigma_1(z)$, and $\Sigma_2(z)$ analytic everywhere in the plane except along the cut (a,b) where Σ_1 and Σ_2 satisfy the conditions,

$$\Sigma_1^+(x) = -e^{(\nu+\lambda)i}\ \frac{\rho}{\sigma}\ \Sigma_1^-(x) + k_1(x)\ , \tag{4.5}$$

$$\Sigma_2^+(x) = -\ e^{(\lambda-\nu)i}\ \frac{\rho}{\sigma}\ \Sigma_2^-(x) + k_2(x)\ , \tag{4.6}$$

$$\Sigma_1(z) = \Sigma_2(z) = O(|z|^{-1}) \quad \text{as} \quad z \to \infty\ .$$

and $\nu, \lambda, \rho, \sigma, k_1(x)$ and $k_2(x)$ are given by:

$$\nu = \tan^{-1}\left[\frac{2\sin\mu\pi\sqrt{\alpha\beta\gamma\delta}}{\delta\alpha - \beta\gamma}\right] \tag{4.7}$$

$$\lambda = \tan^{-1}\left[-\left(\frac{\alpha\delta + \gamma\beta}{\alpha\delta - \gamma\beta}\right)\tan\mu\pi\right] \tag{4.8}$$

$$\rho = [(\delta\alpha - \beta\gamma)^2 + 4\sin^2\mu\pi(\alpha\beta\gamma\delta)]^{\frac{1}{2}} \tag{4.9}$$

$$\nu = [(\alpha\delta)^2 + (\gamma\beta)^2 - 2\alpha\delta\gamma\beta\cos 2\mu\pi]^{\frac{1}{2}} \tag{4.10}$$

where

$$K(x) = \begin{bmatrix} k_1(x) \\ \\ k_2(x) \end{bmatrix} = \left[P(x)A^{-1}(x)\right]\begin{bmatrix} F_1(x) \\ \\ F_2(x) \end{bmatrix}. \tag{4.11}$$

For example, the solution of the coupled pair of equations

$$\int_0^x \frac{\varphi_1(t)dt}{(x-t)} + \int_x^1 \frac{\varphi_2(t)dt}{(t-x)} = f_1, \qquad x \in (-1,1) \tag{4.12}$$

and

$$\int_x^1 \frac{\varphi_1(t)dt}{(t-x)^{\frac{1}{2}}} + \int_0^x \frac{\varphi_2(t)dt}{(t-x)^{\frac{1}{2}}} = f_2 \qquad x \in (-1,1) \tag{4.13}$$

(with f_1 and f_2 constant)

require the determination of the sectionally analytic functions $\Sigma_1(z)$, $\Sigma_2(z)$ with Σ_1 and Σ_2 vanishing at ∞ and satisfying the boundary conditions,

$$\Sigma_1^+(x) = -\Sigma_1^-(x) + k_1, \qquad x \in (-1,1) \tag{4.14}$$

$$\Sigma_2^+(x) = \Sigma_2^-(x) + k_2, \qquad x \in (-1,1) \tag{4.15}$$

where k_1 and k_2 are the complex constants

$$k_1 = (1-i)(f_1+f_2), \quad k_2 = -(1+i)(f_1-f_2) .$$

It is a simple matter to demonstrate that the solutions to the above Riemann Hilbert problems are given by

$$\Sigma_1(z) = \frac{1}{2\pi i}(1+i)(f_1+f_2)\left\{\frac{z}{\sqrt{(z+1)(z-1)}} - 1\right\} \qquad (4.16)$$

and

$$\Sigma_2(z) = -\frac{1}{2\pi i}(1+i)(f_1-f_2)\ \ln\left[\frac{1-z}{1+z}\right] \qquad (4.17)$$

so that

$$\Phi_1(z) = -i[\Sigma_1(z) + \Sigma_2(z)]$$

and

$$\Phi_2(z) = -i[\Sigma_2(z) - \Sigma_1(z)] .$$

The unknown functions φ_1 and φ_2 are then determined from the Abel equations (2.7) and (2.8) where $R(x) = \sqrt[4]{(1-x^2)}$. One can see that even in the simple case (4.12) and (4.13) the results are quite complicated. Fortunately, for applications, one is usually only interested in $\Phi_1(z)$ and $\Phi_2(z)$. For problems in the theory of elasticity, [see Lowengrub [3]] the stress field is expressed in terms of these two sectionally analytic functions.

Secondly, we consider the particular case of (3.1) and (3.2) with

$$\mu = \tfrac{1}{2}, \quad \alpha(x) = \beta, \quad \beta(x) = \frac{\alpha}{x}, \quad \gamma(x) = \alpha \quad \text{and} \quad \delta(x) = -\beta/x$$

where α and β are constants. We show that our methods

yield the same result as in Lowengrub [3]. We consider the coupled system

$$\hat{A}(x)\Phi^{+}(x) = -\hat{B}(x)\Phi^{-}(x) + \hat{F}(x), \qquad -1 < x < 1$$

where $A(x)$, $B(x)$, and $F(x)$ are given in (3.14a) while $\hat{A}$, $\hat{B}$ and $\hat{F}$ are defined by (3.16). It follows that the condition $\det A(x) \neq 0$ implies that $\beta^2 - \alpha^2 \neq 0$. In addition, if $0 < x < 1$ we have

$$G(x) = -\frac{x}{\beta^2 - \alpha^2}\begin{bmatrix} -\left(\frac{\beta^2+\alpha^2}{x}\right) & \frac{2i\alpha\beta}{x^2} \\ -2i\alpha\beta & -\left(\frac{\beta^2+\alpha^2}{x}\right) \end{bmatrix} \tag{4.18}$$

$$P(x) = \begin{pmatrix} \frac{i}{x} & -1 \\ \frac{i}{x} & 1 \end{pmatrix}, \tag{4.19}$$

$$P(x)T(x)P^{-1}(x) = -\frac{1}{x}\begin{pmatrix} (\beta-\alpha)^2 & 0 \\ 0 & (\beta+\alpha)^2 \end{pmatrix} \tag{4.20}$$

while if $-1 < x < 0$,

$$\overline{P(-x)}\ \overline{T(-x)}\ \overline{P^{-1}(-x)} = \frac{1}{x}\begin{pmatrix} (\beta-\alpha)^2 & 0 \\ 0 & (\beta+\alpha)^2 \end{pmatrix} \tag{4.21}$$

and $\det \overline{A(-x)} = \frac{1}{x}(\beta^2 - \alpha^2)$.

Thus, the uncoupled Riemann boundary value problem becomes: determine two sectionally analytic functions $\Sigma_1(z)$ and $\Sigma_2(z)$

that vanish at infinity and along the cut $-1 \le x \le 1$ satisfy the boundary conditions

$$\begin{aligned} \Sigma_1^+(x) &= \left(\frac{\beta-\alpha}{\beta+\alpha}\right) \Sigma_1^-(x) + k_1(x) , \qquad -1 < x < 1 \\ \Sigma_2^+(x) &= \left(\frac{\beta+\alpha}{\beta-\alpha}\right) \Sigma_2^-(x) + k_2(x) , \qquad -1 < x < 1 \end{aligned} \tag{4.22}$$

This is in complete agreement with Lowengrub [3], where the solution to the original pair of Abel equations is given along with an application to elasticity.

§5. Application to dual relations.

In what follows, use will be made of certain well known operators of fractional integration, differentiation and Hankel transforms which were introduced by Erdelyi and Sneddon [1]. Let $S_{\eta,\alpha}$ denote the Hankel transform

$$S_{\eta,\alpha}\{f(\xi);x\} = 2^{\alpha}x^{-\alpha}\int_0^{\infty} \xi^{1-\alpha}f(\xi)J_{2\eta+\alpha}(\xi x)d\xi , \tag{5.1}$$

and for $\alpha > 0$ define the fractional integral operators $I_{\eta,\alpha}$ by

$$I_{\eta,\alpha}\{f(\xi);x\} = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\int_0^{x} (x^2-\xi^2)^{\alpha-1}\xi^{2\eta+1}f(\xi)d\xi \tag{5.2}$$

$$K_{\eta,\alpha}\{f(\xi);x\} = \frac{2x^{2\eta}}{\Gamma(\alpha)}\int_x^\infty (\xi^2-x^2)^{\alpha-1}\xi^{-2\alpha-2\eta+1}f(\xi)d\xi. \tag{5.3}$$

For $\alpha < 0$ the fractional derivatives $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ are defined to be the formal inverses of the operators $I_{\eta+\alpha,-\alpha}$ and $K_{\eta+\alpha,-\alpha}$ respectively. Frequent use will be made of the well known identities relating these operators. [See Sneddon [7], page 274]

We shall consider systems of dual relations of the form

$$\begin{aligned}
&\int_0^\infty [aA(\xi) + bB(\xi)]\xi^{-2\alpha}J_\mu(\xi x)d\xi = f_1(x) && 0 < x < 1\\
&\int_0^\infty [cA(\xi) + dB(\xi)]\xi^{-2\beta}J_\nu(\xi x)d\xi = f_2(x) &&\\
&\int_0^\infty A(\xi)J_\mu(\xi x)d\xi = 0 && \\
&\int_0^\infty B(\xi)J_\nu(\xi x)d\xi = 0\ , && 1 < x < \infty
\end{aligned} \tag{5.4}$$

where a,b,c and d are constants.

Systems similar to (1) have been discussed by various authors. Closed forms solutions have been obtained by Lowengrub and Sneddon [4] for the cases $\alpha = \beta = -\frac{1}{2}$, $\mu = 0$, $\nu = 1$ and $\alpha = \beta = -\frac{1}{2}$, $\mu = \frac{1}{2}$, $\nu = -\frac{1}{2}$. In the former instance, the system (1) was reduced to a system of Carlemann type singular integral equations whereas in the latter the generalized Abel system

(3.1, 3.2) was obtained. Westmann used similar techniques to construct solutions for $\alpha = \beta$ and $\mu = \nu = 2$, whereas Erdogan [2] presented a method for reducing (1) to an infinite system of algebraic equations.

The procedures developed for treating special cases of (5.4) have mostly been ad hoc. We shall indicate a more systematic approach for analyzing (5.4) which in certain cases reduces to the generalized Abel system (3.1, 3.2).

In operator notation the system (5.4) becomes

$$S_{\frac{\mu}{2}-\alpha,2\alpha}\{[a\varphi(\xi)+b\Psi(\xi)];x\} = F_1(x), \qquad 0<x<1 \tag{5.6}$$

$$S_{\frac{\nu}{2}-\beta,2\beta}\{[c\varphi(\xi)+b\Psi(\xi)];x\} = F_2(x), \qquad 0<x<1 \tag{5.7}$$

$$S_{\frac{\mu}{2},0}\{\varphi(\xi);x\} = g_1(x)H(1-x), \qquad x>1 \tag{5.8}$$

$$S_{\frac{\nu}{2},0}\{\Psi(\xi);x\} = g_2(x)H(1-x), \qquad x>1 \tag{5.9}$$

where $\varphi(\xi) = A(\xi)/\xi$, $\Psi(\xi) = B(\xi)/\xi$, $F_1(x) = 2^{2\alpha}f_1(x)/x^{2\alpha}$, $F_2(x) = 2^{2\beta}f_2(x)/x^{2\beta}$ and $g_1(x)$ and $g_2(x)$ are unknown functions for $0<x<1$.

Formal inversion of (5.8) and (5.9) yields

$$\varphi(\xi) = S_{\frac{\mu}{2},0}\{g_1(x);\xi\}$$

$$= S_{\frac{\mu}{2},0} \circ K_{\frac{\mu}{2},\gamma}[k_1]$$

$$= 2^{\gamma}\xi^{-\gamma}\int_0^1 x^{1-\gamma} J_{\mu+\gamma}(x\xi)k_1(x)\,dx \qquad (5.10)$$

and

$$\Psi(\xi) = S_{\frac{\nu}{2},0}\{g_2(x);\xi\}$$

$$= S_{\frac{\nu}{2},0} \circ K_{\frac{\nu}{2},\delta}[k_2]$$

$$= S_{\frac{\nu}{2},\delta}[k_2]$$

$$= 2^{\delta}\xi^{-\delta}\int_0^1 x^{1-\delta} k_2(x) J_{\nu+\delta}(x\xi)\,dx \qquad (5.11)$$

where $k_1(x) = K_{\frac{\mu}{2}+\gamma,-\gamma}[g_1]$,

$k_2(x) = K_{\frac{\nu}{2}+\delta,-\delta}[g_2]$,

and γ and δ are parameters to be specified later. The manipulations involved in (5.10) and (5.11) are well known and may be found in Sneddon [7].

Substitution of (5.10) and (5.11) into (5.6) and (5.7), followed by an application of appropriate fractional transforms, yields

$$I_{\frac{\mu}{2}+\alpha,\lambda}\, S_{\frac{\mu}{2}-\alpha,2\alpha}\{[a\varphi(\xi)+b\Psi(\xi)];x\} = \hat{F}_1(x)$$

$$= a\, S_{\frac{\mu}{2}-\alpha,\lambda+2\alpha} \circ S_{\frac{\mu}{2},\gamma}[k_1] + b\, S_{\frac{\mu}{2}-\alpha,\lambda+2\alpha} \bullet S_{\frac{\nu}{2},\delta}[k_2] \tag{5.12}$$

and

$$I_{\frac{\nu}{2}+\beta,\rho} \circ S_{\frac{\nu}{2}-\beta,2\beta}\{[c\varphi(\xi)+d\Psi(\xi)];x\} = \hat{F}_2(x)$$

$$= c\, S_{\frac{\nu}{2}-\beta,\rho+2\beta} \circ S_{\frac{\mu}{2},\gamma}[k_1] + d\, S_{\frac{\nu}{2}-\beta,2\beta} \circ S_{\frac{\nu}{2},\delta}[k_2] \tag{5.13}$$

where $\hat{F}_1(x) = I_{\frac{\mu}{2}+\alpha,\lambda}[F_1]$,

$\hat{F}_2(x) = I_{\frac{\nu}{2}+\beta,\rho}[F_2]$

and λ and ρ are parameters as yet unspecified.

There are two trivial cases of (5.12) and (5.13). If $\mu=\nu$, let $\gamma=\delta$, $\lambda=-\alpha$ and $\rho=-\beta$, then these relations become

$$a\, S_{\frac{\mu}{2}-\alpha,\alpha} \bullet S_{\frac{\mu}{2},\gamma}[k_1] + b\, S_{\frac{\mu}{2}-\alpha,\alpha} \bullet S_{\frac{\mu}{2},\gamma}[k_2] = \hat{F}_1(x)$$

$$= a\, K_{\frac{\mu}{2}-\alpha,\alpha+\gamma}[k_1] + b\, K_{\frac{\mu}{2}-\alpha,\alpha+\gamma}[k_2] \tag{5.14}$$

and

$$c\, S_{\frac{\mu}{2}-\beta,\beta} \bullet S_{\frac{\mu}{2},\gamma}[k_1] + d\, S_{\frac{\mu}{2}-\beta,\beta} \bullet S_{\frac{\mu}{2},\gamma}[k_2] = \hat{F}_2(x)$$

$$= c\, K_{\frac{\mu}{2}-\beta,\beta+\gamma}[k_1] + d\, K_{\frac{\mu}{2}-\beta,\beta+\gamma}[k_2]. \tag{5.15}$$

If we invert relations (5.14) and (5.15), one obtains a simple algebraic system for k_1 and k_2

A second trivial case results if $\frac{\nu-\mu}{2} + \alpha - \beta = 0$, since by choosing $\lambda + \alpha = \rho + \beta = \gamma + \alpha = \delta + \beta = 0$ we obtain

$$a\, S_{\frac{\mu}{2}-\alpha,\alpha} \circ S_{\frac{\mu}{2},-\alpha}[k_1] + b\, S_{\frac{\mu}{2}-\alpha,\alpha} \circ S_{\frac{\nu}{2},-\beta}[k_2] = \hat{F}_1(x)$$

$$= ak_1 + b\, I_{\frac{\nu}{2},\alpha-\beta}[k_2] \tag{5.16}$$

and

$$c\, S_{\frac{\nu}{2}-\beta,\beta}\, S_{\frac{\mu}{2},-\alpha} + d\, S_{\frac{\nu}{2}-\beta,\beta}\, S_{\frac{\nu}{2},-\beta}[k_2] = \hat{F}_2(x)$$

$$= c\, I_{\frac{\mu}{2},\beta-\alpha}[k_1] + d\, k_2. \tag{5.17}$$

Application of $I_{\frac{\nu}{2},\alpha-\beta}$ to (5.17) yields

$$c\, k_1 + d\, I_{\frac{\nu}{2},\alpha-\beta}[k_2] = I_{\frac{\nu}{2},\alpha-\beta}[\hat{F}_2]\ . \tag{5.18}$$

It is now a simple matter to solve (5.16) and (5.18) for k_1 and k_2.

In general the system (5.12) and (5.13) cannot be solved so easily. However, it may be simplified in either of two ways. If, for λ, γ, ρ and δ, we choose $\gamma = -\alpha$, $\delta = -\beta$, $\rho = \frac{\mu-\nu}{2} - \beta$ and $\lambda = \frac{\nu-\mu}{2} - \alpha$ then the system becomes

$$a\, I_{\frac{\mu}{2},\frac{\nu-\mu}{2}}[k_1] + b\, K_{\frac{\mu}{2}-\alpha,\frac{\nu-\mu}{2}+(\alpha-\beta)}[k_2] = \hat{F}_1 \qquad (5.19)$$

$$c\, K_{\frac{\nu}{2}-\beta,\frac{\mu-\nu}{2}+(\beta-\alpha)}[k_1] + d\, I_{\frac{\nu}{2},\frac{\mu-\nu}{2}}[k_2] = \hat{F}_2. \qquad (5.20)$$

Whereas by choosing $\gamma = \frac{\nu-\mu}{2} - \beta,$, $\delta = \frac{\mu-\nu}{2} - \alpha,$ $\rho = -\beta$ and $\lambda = -\alpha$ we obtain

$$a\, K_{\frac{\mu}{2}-\alpha,\frac{\nu-\mu}{2}+(\alpha-\beta)}[k_1] + b\, I_{\frac{\nu}{2},\frac{\mu-\nu}{2}}[k_2] = \hat{F}_1 \qquad (5.21)$$

$$c\, I_{\frac{\mu}{2},\frac{\nu-\mu}{2}}[k_1] + d\, K_{\frac{\nu}{2}-\beta,\frac{\mu-\nu}{2}+(\beta-\alpha}[k_2] = \hat{F}_2\ . \qquad (5.22)$$

The systems (5.19) - (5.22) may be regarded as generalized Abel systems. In most applications $\alpha = \beta$ and hence we shall consider only this case for the remainder of this section. It should be observed that when $\nu - \mu$ is an even integer the operators appearing in (5.19) - (5.22) are not of fractional order. It is then a simple matter to reduce either of the systems (5.19), (5.20)
or (5.21), (5.22) to a single linear ordinary differential equation. In particular, for the case considered by Westmann [8] (i.e. $\nu = \mu + 2$) the system (5.19), (5.20) reduces to a simple first order linear differential equation.

If $\mu - \nu$ is not an even integer then the operators are

of fractional order, with both fractional integrals and fractional derivatives appearing in both systems. However, in certain cases it is possible to reduce the systems (5.19) - (5.22) to Abel systems of the form (3.1) and (3.2). We might note that if $\mu - \nu$ is one . the Abel system obtained will contain only operators of order 1/2, whereas in general, operators of different order will occur within the same system. Provided neither of the unknown functions appears in operators of different order, a straight forward extension of the technique presented in section 3 will treat such systems.

As an example, consider the system (1) with $\nu = 1$, $\mu = 0$, $\alpha = \beta = \frac{1}{2}$. If we let $\lambda = \rho = -\frac{1}{2}$, $\delta = -1$ and $\gamma = 0$ we obtain

$$a\, K_{-\frac{1}{2},\frac{1}{2}}[k_1] + b\, I_{\frac{1}{2},-\frac{1}{2}}[k_2] = \hat{F}_1(x) \tag{5.23}$$

$$c\, I_{0,\frac{1}{2}}[k_1] + d\, K_{0,-\frac{1}{2}}[k_2] = \hat{F}_2(x)\ . \tag{5.24}$$

Since,

$$I_{\frac{1}{2},-\frac{1}{2}}[k_2] = \int_0^x (x^2 - t^2)^{-\frac{1}{2}} \frac{d}{dt}[tk_2(t)]dt$$

$$K_{0,-\frac{1}{2}}[k_2] = -\int_x^1 (t^2 - x^2)^{-\frac{1}{2}} \frac{d}{dt}[tk_2(t)]dt,$$

$k_1(t) = g_1(t)$ and

$$k_2(t) = K_{-\frac{1}{2},1}[g_2] = 2t^{-1}\int_t^1 g_2(u)du,$$

the system (5.23), (5.24) yields the generalized Abel system,

$$\frac{a}{x}\int_x^1 (t^2-x^2)^{-\frac{1}{2}} t g_1(t)\,dt - b\int_0^x (x^2-t^2)^{-\frac{1}{2}} g_2(t)\,dt = \frac{\Gamma(\frac{1}{2})}{2}\hat{F}_1(x) \qquad (5.25)$$

$$0 < x < 1$$

$$\frac{c}{x}\int_0^x (x^2-t^2)^{-\frac{1}{2}} t g_1(t)\,dt + d\int_x^1 (t^2-x^2)^{-\frac{1}{2}} g_2(t)\,dt = \frac{\Gamma(\frac{1}{2})}{2}\hat{F}_2(x) \qquad (5.26)$$

$$0 < x < 1 \; .$$

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